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A COMPLETION PROBLEM FOR FINITE AFFINE PLANES

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A partial affine plane (PAP) of order n is an n^2-set S of points together with a collection of n-subsets of S called lines such that any two lines meet in at most one point. We obtain conditions under which a PAP with nearly n^2+n lines can be completed to an affine plane by adding lines. In particular, we make use of Bruck's completion condition for nets to show that certain PAP's with at least $n^2+n-\sqrt{n}$ can be completed and that for $n\neq 3$ any PAP with n^2+n-2 lines can be completed.

1. Introduction

In this paper we study incidence structures with the parameters of a finite affine plane missing some lines. A *partial affine plane* (PAP) of order n is a pair $(S, \mathcal{A})$ where S is an n^2-set of *points* and $\mathcal{A}$ is a collection of n-subsets of S called *lines* such that any two lines meet in at most one point. Two points are said to be *joined* if they lie on a common line. By counting joined pairs of points one obtains the inequality $b\leqq n^2+n$, where b is the number of lines. A PAP with $b=n^2+n$ is an *affine plane*. We say that a PAP $(S, \mathcal{A})$ *can be completed* if there exists an affine plane $(S, \mathcal{A}')$ with $\mathcal{A}\subset\mathcal{A}'$.

We will show that under certain conditions a PAP with nearly n^2+n lines can be completed. This problem was suggested to the author by Richard M. Wilson. Specifically he asked whether a PAP with $b\geqq n^2+2$ can always be completed. The following example shows that this would be best possible. Let $(S, \mathcal{A})$ be an affine plane of order $n\geqq 3$. Let $A\in\mathcal{A}, p\in A, q\in S\setminus A$, $A'=(A\setminus\{p\})\cup\{q\}$, $\mathcal{A}'=(\mathcal{A}\setminus\{A\})\cup\cup\{A'\}$, and $\mathcal{B}=\{B\in\mathcal{A}: q\in B \text{ and } \emptyset\neq B\cap A\neq\{p\}\}$. Then $(S, \mathcal{A}'\setminus\mathcal{B})$ is a PAP with $b=n^2+1$ which cannot be completed.

The author [3] previously studied the completion problem for structures called partial projective planes, which have the parameters of a finite projective plane missing some lines. In the next section we apply known results on partial projective planes to prove two elementary results on PAP's. Bruck [2] has given conditions under which a net can be completed. In section 3 we apply one of Bruck's theorems to show that certain PAP's with $b>n^2+n-\sqrt{n}$ can be completed. We then show that for $n\geqq 4$ any PAP with $b=n^2+n-2$ can be completed.

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2. A completion condition and uniqueness

Let $(S, \mathscr{A})$ be a PAP of order n. The number of lines containing a given point p is called the *valence* of p, denoted $\mathrm{val}(p)$. Since p is joined to exactly $(n-1)$ $\mathrm{val}(p)$ points, we have $\mathrm{val}(p) \leqq n+1$, with equality if and only if p is joined to every other point. We say that lines A and B are *parallel* if $A=B$ or $A \cap B = \emptyset$.

Theorem 2.1. *Let $(S, \mathscr{A})$ be a PAP of order n.*
(i) *If $(S, \mathscr{A})$ can be completed, then parallel is an equivalence relation on $\mathscr{A}$.*
(ii) *If $b > n^2$ and parallel is an equivalence relation on $\mathscr{A}$, then $(S, \mathscr{A})$ can be completed.*

Proof. Assertion (i) is a direct consequence of the well-known fact that parallel is an equivalence relation on the set of lines of an affine plane.

Assume that $b > n^2$ and that parallel is an equivalence relation on $\mathscr{A}$. The average valence of a point, given by bn/n^2, exceeds n. Hence some point p has valence $n+1$. The lines through p belong to distinct equivalence classes. Any other line meets only n of these lines, so is parallel to one of them. Thus there are exactly $n+1$ equivalence classes. We now form a new incidence structure by adding $n+1$ new points, each incident with all lines of one equivalence class, and adding one new line incident with all the new points. The result is a partial projective plane of order n having at least n^2+2 lines. It can be embedded in a projective plane of order n (see [3]). Removing the $n+1$ new points yields the desired affine plane. ∎

The result on partial projective planes needed in the proof of Theorem 2.1 (ii) was first proved by Vanstone [4] in the dual setting. In view of Theorem A of [3] it may be possible to weaken the inequality $b > n^2$ in the hypothesis. However, note that adding $n-1$ points to a projective plane of order $n-1$ yields a PAP of order n with $b = n^2 - n + 1$ which cannot be completed, and in which parallel is an equivalence relation. Theorem B of [3] may easily be applied to obtain the following uniqueness result.

Theorem 2.2. *A PAP $(S, \mathscr{A})$ of order n with $b > n^2 - n$ can be completed in at most one way; i.e., if $(S, \mathscr{A} \cup \mathscr{B})$ and $(S, \mathscr{A} \cup \mathscr{C})$ are affine planes and $\mathscr{A} \cap \mathscr{B} = \emptyset = \mathscr{A} \cap \mathscr{C}$, then $\mathscr{B} = \mathscr{C}$.* ∎

Theorem 2.2 is best possible. To see this let p, q be two points in an affine plane $(S, \mathscr{A} \cup \mathscr{B})$, where $\mathscr{B}$ is the set of $2n$ lines containing p or q but not both. Replacing p by q and q by p in the lines of $\mathscr{B}$ yields a new affine plane $(S, \mathscr{A} \cup \mathscr{C})$.

3. Completion of the complement of $\sqrt{n}$ parallel or concurrent lines

The proofs in this section require a theorem of Bruck on nets. A *net* N of order n, degree k may be defined as a PAP with $b = kn$ having k parallel classes, where a parallel class is a set of n parallel lines. The *deficiency* of N is defined to be $d = n+1-k$. An affine plane is then a net of deficiency 0. A *transversal* of N is a set of n points no two of which are joined. If A is a transversal of $N = (S, \mathscr{A})$, then $(S, \mathscr{A} \cup \{A\})$ is a PAP. More generally, if $\mathscr{B}$ is a set of transversals any two of which meet in at most one point, then $(S, \mathscr{A} \cup \mathscr{B})$ is a PAP. Bruck [2] showed that if $d < \sqrt{n}+1$, then any two transversals meet in at most one point. Thus he established

Theorem 3.1. (Bruck). *Let N be a net of order n, deficiency $d < \sqrt{n}+1$ and let $\mathscr{B}$ be the set of all transversals of N. Then $|\mathscr{B}| \leqq dn$, with equality if and only if N can be completed (in which case $(S, \mathscr{A} \cup \mathscr{B})$ is an affine plane).* ▮

Let P be a PAP of order n with $b = n^2+n-e$. Let v_i be the number of points of valence i, $0 \leqq i \leqq n+1$. We are interested in two special "valence distributions", which would occur if P were obtained from an affine plane by removing e parallel or concurrent lines:

(1) $$v_{n+1} = n^2 - en, \quad v_n = en \quad (e \leqq n),$$

(2) $$v_{n+1} = (n+1-e)(n-1), \quad v_n = e(n-1), \quad v_{n+1-e} = 1 \quad (e \leqq n+1).$$

Distributions (1) and (2) are extreme in some sense. If every point of P has valence at least n, then P has distribution (1). We now show that $n+1-e$ is the smallest valence which can occur in P, and that if this valence occurs, then P has distribution (2). Let $\operatorname{val}(q) = r$. The $r(n-1)$ points joined to q have valence at most $n+1$ and the $(n+1-r)(n-1)$ points unjoined to q have valence at most n. The sum of the valences of all points is bn, so we have $r + r(n-1)(n+1) + (n+1-r)(n-1)n \geqq bn$, or $r \geqq n+1-e$.

Theorem 3.2. *Let P be a PAP with $b = n^2+n-e$. If $e < \sqrt{n}+1$ and some point has valence $n+1-e$, then P can be completed.*

Proof. Let $\operatorname{val}(q) = n+1-e$. From the preceding remarks we see that the points joined to q have valence $n+1$ and that those unjoined to q have valence n. Let A be a line through q and let p be a point not on A. Then p is on exactly n lines not containing q. Of these, $n-1$ join p to the $n-1$ points of valence $n+1$ on A. Thus p is on a unique line parallel to A. Therefore each line A through q defines a parallel class of n lines. The lines of these parallel classes form a net N of order n, degree $n+1-e$. The remaining $en-e$ lines of P are transversals of N. To complete the proof it suffices to exhibit e distinct transversals of N which are not lines of P, since N would then have en transversals and Theorem 3.1 is in force.

We claim that each set T_p, consisting of a point p of valence n together with the $n-1$ points unjoined to p, is a transversal of N. To see this let $\operatorname{val}(p) = n$ and let $A_1, A_2, \ldots, A_n$ be a parallel class from N with $p \in A_1$. Each line through p other than A_1 meets all of $A_2, \ldots, A_n$. Therefore $|A_i \cap T_p| = 1$ for each i, and T_p is a transversal of N. Since each of the $e(n-1)$ points of valence n is in at least one set T_p and q is in every set T_p, there are at least e distinct sets T_p and the proof is complete. ▮

Theorem 3.3. *Let P be a PAP with $b = n^2+n-e$. If $e < \sqrt{n}$ and some line contains only points of valence $n+1$, then P can be completed.*

Proof. Let A be a line containing only points of valence $n+1$. Every point p is joined to all points of A. Hence $\operatorname{val}(p) \geqq n$ and $\operatorname{val}(p) = n+1$ if and only if p is on a line parallel to A. Therefore P has valence distribution (1). Exactly $n-e$ lines contain only points of valence $n+1$. Each of the remaining n^2 lines contains $n-e$ points of valence $n+1$ and e points of valence n.

Let P' be the dual of P. A line of P' has size n or $n+1$; we call it a short or long line accordingly. Every point of P' has valence n. There are $n-e$ special points, each incident with n long lines. Each of the remaining n^2 points is on $n-e$ long lines and e short lines. The long lines form a net N of order n, degree $n-e$ on these n^2 points. The short lines are transversals of N. For each point p of N define T_p to be the set consisting of p together with the $n-1$ points unjoined to p. Each of the n lines through p meets all the long lines. It follows that T_p is a transversal of N. Let q be a special point of P' and B be a long line containing q. Any point of N not on B is unjoined to exactly one point of B. Therefore the n sets T_p, $p\in B\setminus\{q\}$, are disjoint transversals of N. Between these n transversals and the short lines, we have $(e+1)n$ transversals of N. By Theorem 3.1 N can be completed to an affine plane by adjoining all of these transversals. In this affine plane there are $n+1$ parallel classes; $n-e$ of these are the parallel classes of N and one parallel class consists of the n disjoint sets T_p. Therefore the en short lines of P' fall into e parallel classes. This means that in P the points of valence n fall into e groups of n points each, no two points within a group being joined. These e groups may be adjoined to P as new lines. Therefore P can be completed. ∎

4. Completion of PAP's with two missing lines

Theorem 4.1. *Let P be a PAP of order n.*
(i) *If $n\geqq 2$ and $b=n^2+n-1$, then P can be completed.*
(ii) *If $n\geqq 4$ and $b=n^2+n-2$, then P can be completed.*

Proof. (i) It is easily seen that P has exactly n points of valence n, no two of which are joined. Therefore P can be completed.

(ii) The remarks preceding Theorem 3.2 show that P has valence distribution (1) or (2). If P has distribution (2), then Theorem 3.2 implies that P can be completed. Assume that P has distribution (1). If $n>4$ and some line contains only points of valence $n+1$, then Theorem 3.1 implies that P can be completed. A tedious analysis shows that P can also be completed if $n=4$ and some line contains only points of valence 5. It now suffices to show that the assumption that every line contains a point of valence n leads to a contradiction.

There are $nv_n=2n^2$ point-line pairs (p, A) such that $p\in A$ and $\mathrm{val}(p)=n$. Since $2n^2<2b$ some line A contains only one point p of valence n. A point of valence n is unjoined to exactly $n-1$ other points, all of which have valence n. Let X denote the set of n points of valence n joined to p, Y the set of n remaining points of valence n, and Z the set of points of valence $n+1$. If $q\in X$, then q is joined to every point of A and is on no line parallel to A. If $q\in Y\cup Z$, then q is on a unique line parallel to A. Therefore the points of $Y\cup Z$ are partitioned by the $n-1$ lines (including A) parallel to A. Each of these $n-1$ lines must contain a point of Y. Therefore all but one of the lines parallel to A contains exactly one point of Y and one contains two points y_1, y_2 in Y. No pair in Y other than (y_1, y_2) is joined. Each point of Y is joined to exactly n points of valence n. Hence every pair (x, y), $x\in X$,

$y \in Y$, is joined except two: (x_1, y_1) and (x_2, y_2), where we may have $x_1 = x_2$. Each point of X other than x_1, x_2 is joined to every point of Y, hence to no other point of X. Hence at most one pair (x_1, x_2) in X is joined. Let $y \in Y \setminus \{y_1, y_2\}$. The line through y parallel to A contains no point of X. The remaining $n-1$ lines through y join y to all n points of X, so some line through y joins a pair of points in X. Therefore at least $n-2$ pairs in X are joined. Since only one pair in X can be joined, we have $n-2 \leqq 1$, which contradicts the hypothesis. ▮

5. Remarks

The problem of finding the largest b for which there exists a PAP of order n (not a prime power) with b lines is a special case of a widely studied packing problem also studied in the setting of constant weight codes (see [1]). Using the notation of [1], Theorem 4.1 implies that if $A(n^2, 2n-2, n) \neq n^2+n$, then $A(n^2, 2n-2, n) \leqq \leqq n^2+n-3$. For $n=6$ the author is unaware of better bounds than $32 \leqq \leqq A(36, 10, 6) \leqq 39$, the lower bound being obtained by adding five points and one line to $PG(2, 5)$.

Obviously any PAP of order 2 can be completed, but already for for $n=3$ there exist *maximal* PAP's with $b=8, 9$ and 10. The cases $b=8$, 10 are obtained from $PG(2, 2)$ and $AG(2, 3)$. The other one is an interesting self-dual configuration.

Some of the ideas presented here go through for (k, v)-packings, systems of k-subsets of a v-set any two of which meet in at most one element. For example, it is not hard to show that if $v \equiv 1$ or $k \bmod k(k-1)$, then any (k, v)-packing with $b = v(v-1)/k(k-1)-1$ can be completed to an $S(2, k, v)$.

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